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2016
MCM/ICM
Model of the Temperature of the Bathtub Water

Abstract

Herein we present a four dimensional heat conduction model in time and space to determine the heat distribution of bath water. Due to the difficulty to determine the boundary condition when a man enters the bathtub, we establish and use our own computer simulation system to solve the problem according to the microscopic theories of heat conduction equations.

We view bath water as an aggregation of many tiny points, heat exchanges among these points. When all the parameters are given, we can determine the minimum flow of the trickle to keep the temperature of bath water as we expected. Through computer simulation, we can calculate the variance of temperature distribution and compare “evenness” in diffirent situations.

Our results illustrate that the best strategy is determined mainly by the superficial area of bath water, the material of the bathtub and the position where the faucet is. Besides, the perfect “evenness” can’t be realized. And man’s motion will contribute to even temperature distribution.

Our simulation results also indicate that all the variables including the shape, volume and material of the tub, the shape, volume and motion of the person will affect the heat distribution of bath water, but varies from variable to variable. When we are going to make a best strategy, we need to take them into consideration.

By analyzing the equilibrium state, we determine the relationship among parameters required to reach an ideal temperature. In our sensitivity analysis, we consider the effect of several impact factors independently. Qualitatively, key findings of the sensitivity analysis are:

- The ambient temperature is one of the most effective factors that alter the ultimate average temperature.
- Both the temperature of bath water in the equilibrium state and the time need to reach the stage of equilibrium are very sensitive to the temperature of the trickle.

These findings are used to produce recommendations to users of the bathtub.

Contents

1 Introduction

1.1 Background

A person fills a bathtub with hot water from a single faucet and settles into the bathtub to cleanse and relax. Unfortunately, the bathtub is not a spa-style tub with a secondary heating system and circulating jets, but rather a simple water containment vessel. After a while, the bath gets noticeably cooler, so the person adds a constant trickle of hot water from the faucet to reheat the bathing water. The bathtub is designed in such a way that when the tub reaches its capacity, excess water escapes through an overflow drain.

Besides, we need notice that the simple water containment vessel is safer than a spa-style tub with a secondary heating system and circulating jets. Especially in developing country, accidents caused by the leakage of electricity when showing occur in higher frequency. Our work is to create a safe and comfortable bathing based on our practical and operational model.

1.2 Context of the Problem

The problems we are confronted with and need to figure out are as follows:

I. Develop a model of the temperature of the bathtub water in space and time to determine the best strategy the person in the bathtub can adopt to keep the temperature even throughout the bathtub and as close as possible to the initial temperature without wasting too much water.

II. Use your model to determine the extent to which your strategy depends upon the shape and volume of the tub, the shape, volume and temperature of the person in the bathtub, and the motions made by the person in the bathtub.

III. If the person used a bubble bath additive while initially filling the bathtub to assist in cleansing, how would this affect your model's results?

IV. In addition to the required one-page summary for your MCM submission, your report must include a one-page non-technical explanation for users of the bathtub that describes your strategy while explaining why it is so difficult to get an evenly maintained temperature throughout the bath water.

1.3 Our understanding of the words

There are three key points of the problem: First, the temperature of water in the bathtub should be kept even. Second, the temperature should be as close to the initial temperature as possible. Third, the strategy should avoid wasting too much water while keeping the temperature in an ideal level. Thus, our model is supposed to calculate the distribution of temperature in the bathtub under different conditions. Moreover, results of the calculation may be affected by many other variables, so we ought to analyze the influence of each variety on our model so that we can determine an optimal strategy.

The approach we use is called control variables method. To test the certain variable's influence, we keep other variables invariant. The best stretagy should consider all possible influences. It doesn't mean all the solutions are the optimal solutions. The best stretagy must exert the max utility.

1.4 Problems analysis and our task

The irregularity of the bathtub's shape adds a lot of difficulties to our work, we are supposed to view the bathtub as an ideal regular body. We sought pictures of bathtubs on the internet, and found that most of bathtubs are cuboid-like. Therefore, we simplified the bathtub as a hollow homogeneous cuboid without a lid.

Additionally, according to requests and our own understanding, we assume the temperature of hot water from the faucet is constant and regard the trickle as an internal heat source. Those preliminary ideal assumptions will make our work much easier and more operational.

First, we consider the simple situation that there is no person in the bathtub. In this situation, the thermal equilibrium equation of bath water is influenced by two factors: the heat releasing from the trickle and the heat loss through evaporation or heat transfer. According to the direction of heat change and its absolute value, we can ascertain whether the bath water is in heat liberation or heat absorption. As for details in heat loss, we consider three main aspects: the heat exchanges between water and the bathtub, the heat transfers from water to air and the evaporation of water. Thus we can establish partial differential equations, find answers and analyze results.

When we understand the mechanism of this simple case, we are ready to consider some complex situations such as the shape or volume of the tub, the shape, volume or temperature of the person in the bathtub, the motions made by the person in the bathtub and the density or concentration of a bubble bath water. We will analyse each conditions respectively and to ascertain how variables affect the heat distribution and the equilibrium temperature.

What's more, we should not only focus on accurate analytic solutions. Accurate analytic formulas sometimes are not easy to derive from partial differential equations, so we can take full advantage of numerical solutions. Since the boundary conditions of a partial difference problem are not easy to determine, we might resort to computer simulation to help us figure out the problem.

Our given task is to develop a model of the temperature of the bathtub water in space and time to determine the best strategy that can keep the temperature even and close to the initial temperature with less wasting of water. Then we need discuss all uncertain variables' effects on our model and give a sensitivity analysis. Additionly, we will also consider some other factors that we possibly meet in our real life to make our results coincide with the real problem.

1.5 Previous Research

To begin with, we search a wide array of journals and papers related to our problem to help us deepen the understanding of our problems. The equations of heat flow describing the transfer of thermal energy and its corresponding initial and boundary conditions are easily formulated according to conservation of heat energy, researchers generally pursue mathematical solutions involving partial differential equations to the answer[1].

However, the basic solutions doesn't work sometimes due to the complexity and uniqueness of the problem. Therefore, researchers need to find a new approach to overcome all the difficulties. Here are some useful previous researches.

For example, Fajie *et al.* applied a nonsingular indirect boundary element method for the solution of three-dimensional heat conduction problems[2]. Piotr D. was aimed to make a numerical and experimental verification of two methods based on the control volume finite element method and the control volume method[3]. Carlomagno *et al*. derived a generalized heat-conduction equation accounting for non-local, non-linear and relaxation effects by means of dynamical non-equilibrium temperature [4]. Gordeliy *et al.* developed an analytical and computational basis for modeling time-dependent effects due to heat conduction and thermoelasticity in heterogeneous media[5]. Ayriyana *et al.* constructed a model of a multilayer device with non-trivial geometrical structure and nonlinear dependencies of thermodynamic material properties at cryogenic temperatures is suggested[6].The methods of solving models of heat conduction, namely analytical and numerical methods, were described in literature[7].

The literatures mentioned above enlightened us to a great extent, helping us work smoothly.

2 Our Model

2.1 Modeling Objectives

Our modeling objectives is to develop a model of the temperature of the bathtub water in space and time to determine the best strategy that can keep the temperature even and close to the initial temperature with less wasting of water. After that, we need discuss all uncertain variables' effects on our model. Additionally, we are supposed to give a sensitivity analysis and discuss the reliability of our model.

In a word, we are going to solve an actual problem that we possibly meet in our daily life by applying methametical models and computer programming.

2.2 Symbols and definitions

The symbols we use and their definitions are shown in table 2-1.

Table 2-1 Main Symbols and their Definitions

Q	the Heat Energy	J
c	the Specific Heat of Water	$J/(kg \cdot K)$
ρ	the Density of Water	kg/m^3
μ	the Temperature of Water	K

V	the Volume of Water	m^3
t	Time Variable	s
k	the Thermal Diffusivity	$W/(m \cdot K)$
‘ Δ	The Change of Variables	
$F(x, y, z, t)$	the Heat Intensity of the Heat Source	
μ_1	the Temperature of Air.	K
Δq	the Heat Passing from Point to Another	J
R	the Heat Resistance,	
Q_e	the Heat Loss of Water by Evaporating	J
β	the Evaporation Coefficient of Water	
S_w	the Superficial Area of Bath Water	m^2
n	the Amount of Points per Unit Area	
p	the Pressure of Water Vapor	Pa

2.3 Problem Space

Breaking down these ambitious goals into something that we could feasibly achieve and realize in only 96 hours was not a simple task. To simplify the problems our model must deal with, we have seen it prudent to delimit a problem-space through the definition of boundary assumptions. In the problem analysis, we have discussed some preliminary ideal assumptions. But only with those assumptions, we still can’t solve the problem because of the complexity of reality.

Here we give a more sufficient assumptions. These assumptions and reasonable explanations are listed as follows:

- The bathtub is a hollow homogeneous cuboid and this cuboid doesn't has a lid.
- Although there is water flowing in and out, the volume of bath water keeps a constant.
- The volume of trickle is much less than that of bath water, so we ignore the influence of water flow turbulence caused by this constant trickle of hot water from the faucet.
- Because the trickle has an initial velocity and its diameter can't be neglected, we simplify the part of trickle entering the bath water as a steel-like cylinder. Although the water is moving, this fictitious steel-like cylinder is fixed. Moreover, the cylinder can be regarded as a constant heat source.
- Heat transfers among water, bathtub and air. There are three basic modes of heat transfer: thermal radiation, conduction and convection. Thermal radiation only dominates when temperature is extremely high. Heat transfer in fluids is primarily by conduction if the fluid velocity is sufficiently small. Consider the small trickle, we only discuss heat conduction in our model.
- The space of room is settled and the room can get sufficient air exchange with outside word.
- The initial heat distribution of bath water is uniform.
- The ambient temperature is a constand.

With those assumptions, we are more likely to solve a complex actual problem and put forward our proposal for problem solutions.

To totally understand the problem better, we build a space rectangular coordinate as shown in figure 1. The red point is the base point, which is the center of cuboid. Our analysis of model and results presented are all based on this coordinate. As shown in the figure 1, $2\delta_1, 2\delta_2, 2\delta_3$ are the width, height and length of a cuboid-like bathtub respectively.

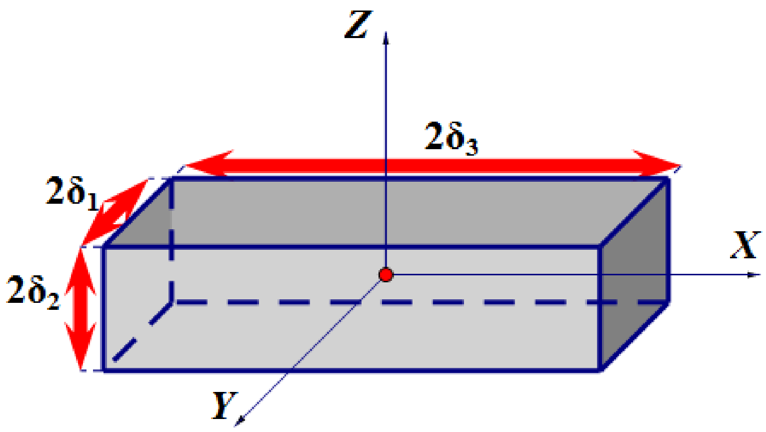

Figure 1 Bathtub in the Space Rectangular Coordinate

2.4 Model Theory

The real problem is concrete but complicated while a model can abstract the important factors that may determine our decision. We are going to establish a mathematic model to simplify and analyze the real problem by applying model theories. In this chapter, we will discuss our model theory by deducing.

2.4.1 Derivation of the Heat Equation in Three Dimensions

This part we begin our derivation by considering any arbitrary subregion Ω of object G. To calculate the heat distribution, we realize the importance of the law of conservation of energy. Conservation of heat energy is summarized by the following word equation:

$$\begin{matrix} \textit{rate of change} \\ \textit{of heat energy} \end{matrix} = \begin{matrix} \textit{heat energy flowing across} \\ \textit{the boundaries per unit time} \end{matrix} + \begin{matrix} \textit{heat energy generated} \\ \textit{inside per unit time} \end{matrix}$$

where the heat energy within an arbitrary subregion Ω is

$$Q = \iiint_{\Omega} c\rho\mu dV \tag{2-1}$$

Thus we have

$$\Delta Q = \int_{t_1}^{t_2} \left[\iiint_{\Omega} c\rho \frac{\partial \mu}{\partial t} dV \right] dt \tag{2-2}$$

where ΔQ is the heat energy change from t_1 to t_2.

According to Fourier's law of heat conduction, the heat ΔQ_1 enters subregion Ω through curved surface S from t_1 to t_2 can be expressed as

$$\Delta Q_1 = \int_{t_1}^{t_2} \iint_S k(x,y,z) \frac{\partial \mu}{\partial n} dS dt \qquad (2\text{-}3)$$

where k is called the thermal diffusivity, n is a unit outward normal to the boundary surface. The sign of ΔQ_1 can either positive or negative.

On the basis of Gauss formula

$$\iiint_\Omega divA dxdydz = \iint_S A \cdot n dS_x \qquad (2\text{-}4)$$

Then we have

$$\Delta Q_1 = \int_{t_1}^{t_2} \left[\iiint_\Omega \left(\frac{\partial}{\partial x}\left(k\frac{\partial \mu}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial \mu}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial \mu}{\partial z}\right)\right) dV \right] dt \qquad (2\text{-}5)$$

Let $F(x,y,z,t)$ be the heat intensity of the heat source. The energy offered by heat source from t_1 to t_2 is

$$\Delta Q_2 = \int_{t_1}^{t_2} \left[\iiint_\Omega F(x,y,z,t) dV \right] dt \qquad (2\text{-}6)$$

Consider conservation of heat energy, we have

$$\Delta Q = \Delta Q_1 + \Delta Q_2 \qquad (2\text{-}7)$$

Due to the arbitrariness of Ω, t_1 and t_2, the equation(2-7) is in equivalence with

$$\frac{\partial \mu}{\partial t} = \alpha^2 \left(\frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} + \frac{\partial^2 \mu}{\partial z^2} \right) + f(x,y,z,t) \qquad (2\text{-}8)$$

where $\alpha^2 = k/c\rho, f = F/c\rho.$.

2.4.2 Conditions for determining solution

The key to solve a partial derivative problem is to determine the initial and boundary conditions. In this part, we will discuss those conditions.

- Initial Condition:

$$\mu(x,y,z,t)\big|_{t=0}=\varphi(x,y,z),\quad (x,y,z)\in G \tag{2-9}$$

In our model, we assume $\varphi(x,y,z)$ is a constant. It indicates that before we open the faucet and add hot water to the bathtub, the heat distribution of bath water is uniform.

- Dirichlet Boundary Condition:

$$\mu\big|_{\Gamma}=g(x,y,z,t),\quad (x,y,z)\in\Gamma,\quad t>0 \tag{2-10}$$

where Γ is the boundary. Note when $g(x,y,z,t)=0$, Object's surfaces are isolated.

- Neumann Boundary Condition:

$$k\frac{\partial\mu}{\partial n}\Big|_{\Gamma}=g(x,y,z,t),\quad (x,y,z)\in\Gamma,\quad t>0 \tag{2-11}$$

Note when $g(x,y,z,t)=0$, Object's surfaces keep a constant temperature.

- D-N Boundary Condition:

$$\left(\frac{\partial\mu}{\partial n}+\frac{k_1}{k}\mu\right)\Big|_{\Gamma}=\frac{k_1}{k}\mu_1(x,y,z,t),\quad (x,y,z)\in\Gamma,\quad t>0 \tag{2-12}$$

where μ_1 is the temperature of medium in touch with water surface. In our case, the medium is air. k_1 is called the heat transfer coefficient. This coefficient is a constant.

2.4.3 Solution by Dimensionality Reduction

We are not hasty in solving the definite problem above, because it is a complicated problem in four dimensions. We attempt to decompose it in low-dimension. Eventually, we have

$$\mu(x,y,z,t)=P(x,t)P(y,t)P(z,t) \tag{2-13}$$

where $P(x,t)$, $P(y,t)$, $P(z,t)$ is the temperature function of definite plate.

2.4.4 Our solution realizing by computer simulation

Although we can get an approach to reduce the dimensionality of equations and use computer to give numerical solutions instead, it still takes us some time to wait before the computer gives us a satisfying answer. Moreover, how to determine the boundary condition when a man enters the bathtub add a lot of difficulties to our work.

Therefore, we are going to solve the equations by computer simulation. We are ready to simulate the process of dynamic heat conduction to get the numerical solutions to equations.

In our problem space, the bathtub is a regular body. Regular datasets are often constructed by an array in three dimensions. Here we use arrays[x][y][z] to present a cuboid bathtub. Vividly speaking, we will get lots of regularly arranged points that can store the information of heat attribute, their location, the points next to them and so on. We notice that heat exchange occurs while there is a temperature difference between two points. Thus we scope the problem in a micro approach.

For the heat source, we set it a fixed temperature; For the rest, we input an initial temperature T . We divide points into three groups: points of heat source, normal points and boundary points, and put different labels on them. First of all, consider the first type of points. If it is a steady heat source, it releases or absorbs heat while keeping temperature constant. In our model, the heat source is the trickle. The trickle can be viewed as an aggregation of many small heat source points. The points' temperature ought to be no lower than that of other kinds of points in order to create a comfortable bathing. The points next to heat points will get their heat and increase in temperature.

Then we discuss the second type of points. If points' temperature next to them is higher, it will absorb heat and increase in temperature. Otherwise, it will release heat and decrease in temperature. The normal points differ from heat source points because normal points themselves will change temperature by others' effect.

As for boundary points, they work like normal points but they will also transfer heat with other medium like air. Boundary points are the bridge of internal and external heat exchange. Through boundary points, substance can either absorb heat from outside or release its heat inside.

As shown in figure 2, F is a boundary point, which is releasing heat to other medium such as air. A is a normal point that receive heat from G and release heat to B at the same time. When the total amount of heat releasing by heat sources equals the amount of heat transferred from boundary points to outside, the system reaches an equilibrium state and keep total heat constant.

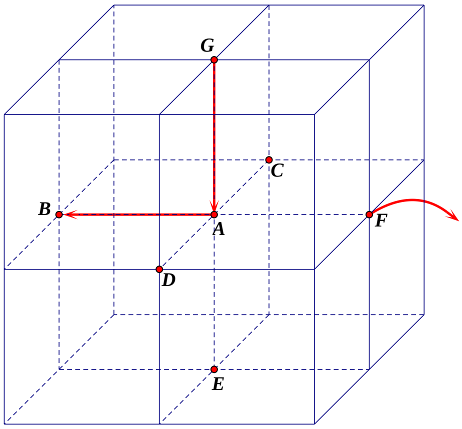

Figure 2

If we understand the micro-mechanism of the heat transfer, than we can use computer simulation to fulfill it. The elaborated process are shown in Figure 3.

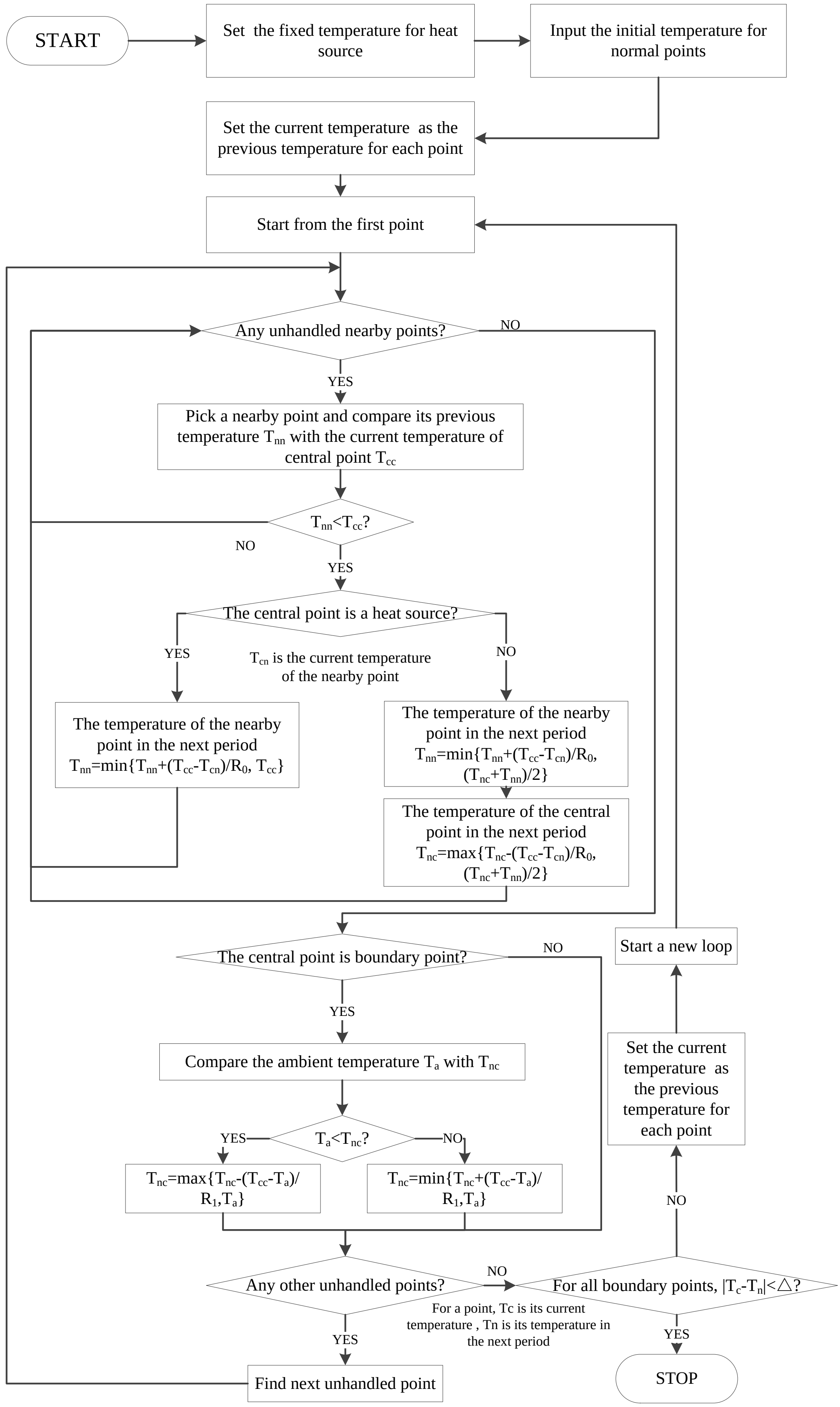
START
Set the fixed temperature for heat source
Input the initial temperature for normal points
Set the current temperature as the previous temperature for each point
Start from the first point
Any unhandled nearby points?
NO
YES
Pick a nearby point and compare its previous temperature T_{nn} with the current temperature of central point T_{cc}
$T_{nn}<T_{cc}$?
NO
YES
The central point is a heat source?
YES
NO
T_{cn} is the current temperature of the nearby point
The temperature of the nearby point in the next period $T_{nn}=\min\{T_{nn}+(T_{cc}-T_{cn})/R_0, T_{cc}\}$
The temperature of the nearby point in the next period $T_{nn}=\min\{T_{nn}+(T_{cc}-T_{cn})/R_0, (T_{nc}+T_{nn})/2\}$
The temperature of the central point in the next period $T_{nc}=\max\{T_{nc}-(T_{cc}-T_{cn})/R_0, (T_{nc}+T_{nn})/2\}$
The central point is boundary point?
NO
YES
Compare the ambient temperature T_a with T_{nc}
$T_a<T_{nc}$?
YES
NO
$T_{nc}=\max\{T_{nc}-(T_{cc}-T_a)/R_1, T_a\}$
$T_{nc}=\min\{T_{nc}+(T_{cc}-T_a)/R_1, T_a\}$
Any other unhandled points?
NO
YES
Find next unhandled point
For all boundary points, $|T_c-T_n|<\triangle$?
For a point, Tc is its current temperature , Tn is its temperature in the next period
YES
STOP
NO
Set the current temperature as the previous temperature for each point
Start a new loop

Figure 3 the Model and Flow Chart

So far, we have constructed the generalization of our way in problem solving. To realize it, Let Δq be the heat passing from point A to point B. What's more, we define a heat resistance R, which is a variable that determined by intrinsic properties of substances. According to the law of thermodynamics, R should be no less than 2. And transferred heat Δq can be expressed as follows

$$\Delta q = \frac{\Delta T}{R} \tag{2-14}$$

The computer will assist us in calculating and eventually gives us numerial solutions.

2.5 Discussion on different situations

2.5.1 Situation when a Person Enters the Bathtub

This part, we consider the condition when a person enters the bathtub. For simplicity, we view the person as a cuboid of constant temperature heat source. When the temperature of man's skin surface is higher than bath water, heat transfers from body to water. Otherwise, the body release heat. Therefore, the new equilibrium equation is

$$\frac{\partial \mu}{\partial t} = \alpha^2 \left(\frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} + \frac{\partial^2 \mu}{\partial z^2} \right) + f\left(x, y, z, t\right) + h(x, y, z, t) \tag{2-15}$$

where $g = G/c\rho$, $G(x, y, z, t)$ is the heat intensity of the body. Additionally, if we let $m\left(x, y, z, t\right) = f\left(x, y, z, t\right) + h(x, y, z, t)$, then we have

$$\frac{\partial \mu}{\partial t} = \alpha^2 \left(\frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} + \frac{\partial^2 \mu}{\partial z^2} \right) + m\left(x, y, z, t\right) \tag{2-16}$$

The formula (2-15) is the same as formula (2-8) in form, so the solution to the formula (2-15) can be the solution to the formula (2-8) as well. So here we are not going to repeat it again.

2.5.2 Situation when the volume of bathtub changes

Based on our solution of computer simulation, we consider the process of unidirectional heat transfer in one dimension. At initial time, all the points are in the same degree as shown in figure. When the point one becomes a constant heat source and its temperature is higher that point two. Definitely, it will transfer heat to point two. The heat transferred from point one to point two is

$$\Delta q_1 = \frac{\Delta T_1}{R} \tag{2-17}$$

And in the same way, the heat that point n can get is

$$\Delta q_{n-1} = \frac{\Delta T_1}{R^n} \tag{2-18}$$

The heat transferred from point n to other medium is

$$\Delta q' = \frac{\Delta T_2}{R'} \tag{2-19}$$

where ΔT_2 is the temperature difference between point n and the outside medium. When $\Delta q_{n-1} = \Delta q'$, the equilibrium state is reached. So the value of n determined by temperature difference and heat resistance.

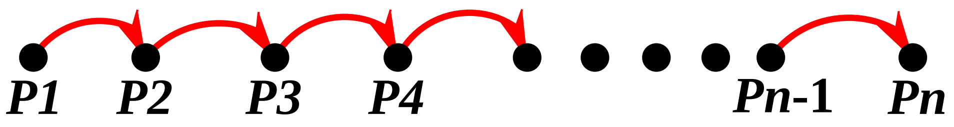

Figure 4 Unidirectional Heat Transfer in One Dimension

It works in the same way in three-dimension. The contribution of volume changes to heat distribution is determined by bath water and air temperature and their heat resistance.

2.5.3 Situation when the shape of bathtub changes

This part, we consider the influence of bathtub's shape on heat distribution. The heat loss due to the heat conduction is

$$Q_3 \propto S_w n_w \frac{\Delta T_w}{R_w} + S_b n_b \frac{\Delta T_b}{R_b} \tag{2-20}$$

where S_w is the area of interface between water and air, S_b is the area of interface between water and bathtub. ΔT_w and ΔT_b is the temperature difference between two substances. n_w, n_b is the amount of points per unit area. R_w and R_b is the heat resistance.

When the area of interface expands, the heat loss through heat conduction will also increase.

Besides, The shape of bathtub will also change the superficial area of bath water.We consider heat loss due to water evaporation. Heat removal by evaporation through the surface of bath water is

$$dQ_e = \beta\left(p_v'' - p_v\right)dS_w \tag{2-21}$$

where Q_e is the heat loss of bath water by evaporating, β is the evaporation coefficient of water, p_v'' is the water vapor pressure of superficial saturated layer, p_v is the partial pressure of water vapor in moist air, S_w is the superficial area of bath water.

The smaller the superficial area is, the less heat water will release. To know to what extent that the shape of bathtub will influence heat distribution, we set volume constant and use computer simulation to calculate the numerical solutions and draw graphs of heat distribution.

So far, we have discussed the influence of the shape and volume of the tub and the shape, volume, and temperature of the person in the bathtub. In most conditions, we control other variables fixed and alter the value of one parameter to know the effect on the heat distribution.

2.5.4 Situation when man’s temperature changes

In this part, we’d like to discuss the influence caused by man’s temperature changes. When man’s temperature changes, one of the parameters in our model change, that is the temperature of the heat source. While keeping the other parameters unchanging, we can get a series of solutions corresponding the different temperatures of man’s body. Then through the analysis of the results, we can know to what extent the man’s temperature will influence heat distribution.

2.5.5 Situation when man’s volume changes

When man’s volume changes, it will alter the number of heat resources and augment the superficial area between shell and water where heat exchange is very exciting. And it will also changer the superficial area between water and air indirectly, which can contribute to the heat distribution as well. We must take all those factors into account in our model, so we can further confirm the influence caused by man's volume.

2.5.6 Situation when man's shape changes

Without any assumption of man's shape, it is very difficult to determine in which degree the shape will influence water heat distribution. Although our simulation can proceed as long as we define the types of points and set a series of initial values of all the parameters, the shape is hard to determine. So in this part, we only talk about some regular geometric bodies such as cuboid, cube, cylinder, ball, trapezoid and other combination of these basic bodies.

2.5.7 Situation considering man's motion

Man's motion is very hard to predict, we must give some assumptions to make the problem easier. We assume, the total volume of water changes periodically caused by man's motion. And the water distribution in each period is the same. So the question is clear. We control other variables fixed and alter the value of the parameter of man's motion periodically. We are going to add a suitable kind of period function to our equations such as trigonometric function, subsection function and so on.

2.5.8 Situation considering a bubble bath

If the person used a bubble bath, it means that the heat resistant between water and air changes. as shown in figure 5, the bubble will hinder the heat transfer from water to air. And a bubble bath will also change the density, concentration and specific heat of bath water. If we can give the initial values, the computer and programming can help us achieve what we are supposed to. Unfortunately, these data are not easy to acquire, so we will give a series of reasonable values base on our understanding and attempt to analysis the sensitivity when initial values change.

In our way of solving the problem. We transform the influence of all those parameters into one single variable R as we assumed before. When we take the heat exchange by evaporation into account, the heat resistance will change. The value of R is mainly determined by two factors, the concentration of bath water and humidity of air.

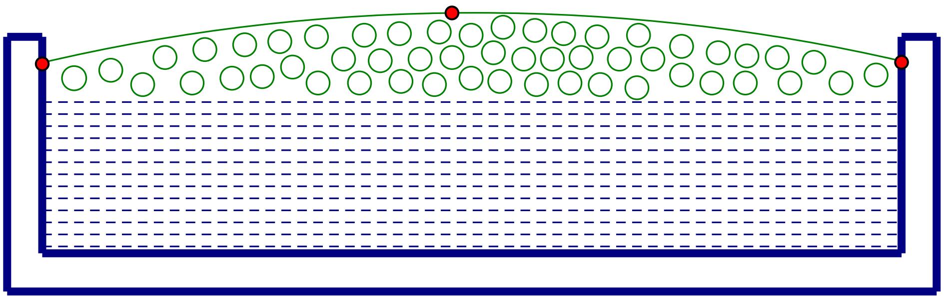

Figure 5 the Bubble Bath Effect

3 Model Implementation and Results

3.1 Parameter setting

In chapter two, we have established a theory model and explained our way of solving equations. To implement the model and get the results, we must set all parameters. Here is the parameters' value we set.

- The length, width and height of a cuboid bathtub is 1.8 m, 0.8 m, 0.6 m respectively.
- Temperature setting: The temperature of the hot trickle is $323K$. The ambient temperature is $298K$. The initial temperature of bath water is $313K$.
- Other parameters like the thermal diffusion are varied from different materials the bathtub use. Even for the same material, according to researcher's work, they have different values. So we know the trend of heat distribution by inputting different values but the values are not corresponding to the reality. In other words, we set some parameters arbitrarily in order to capture the dynamic change of heat distribution but not for an accurate quantitative analysis.

3.2 The temperature of a fixed point varying with time

In this section, we will discuss how the temperature changes before the equilibrium? When the hot trickle enters, its heat cannot immediately pass to all areas of the water. So we test two points, one is the point that's far away from the heat source and the other is the point close to the heat source. There is a big difference in two points. The former point has an obvious process of temperature decreasing. As for the latter one, due to the time of heat transfer is very short and can be ignored.

Therefore, as shown in figure 7, we only find a process of temperature increasing and finally the temperature is definitely close to a certain constant temperature. While in figure 6, we find it obvious that the point has a process of decreasing. Besides, we depict figure 8 to show the average temperature changing with time.

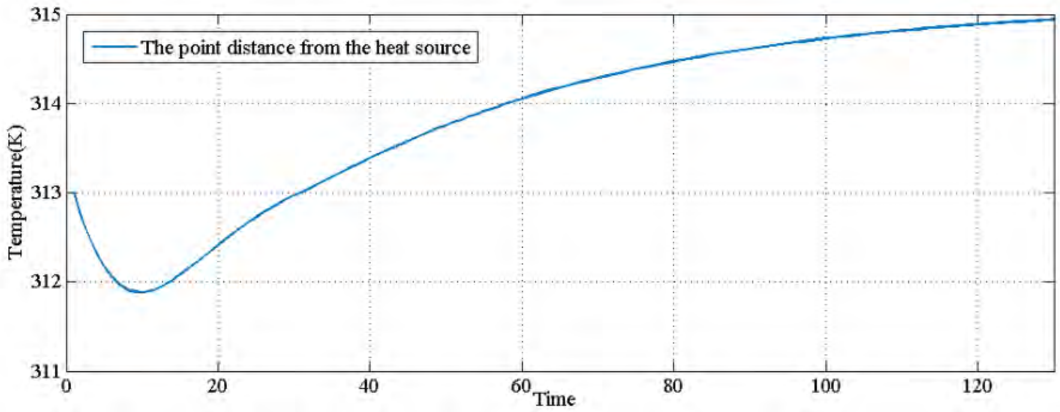

Figure 6 the Temperature Change of the Point Distance from the Trickle

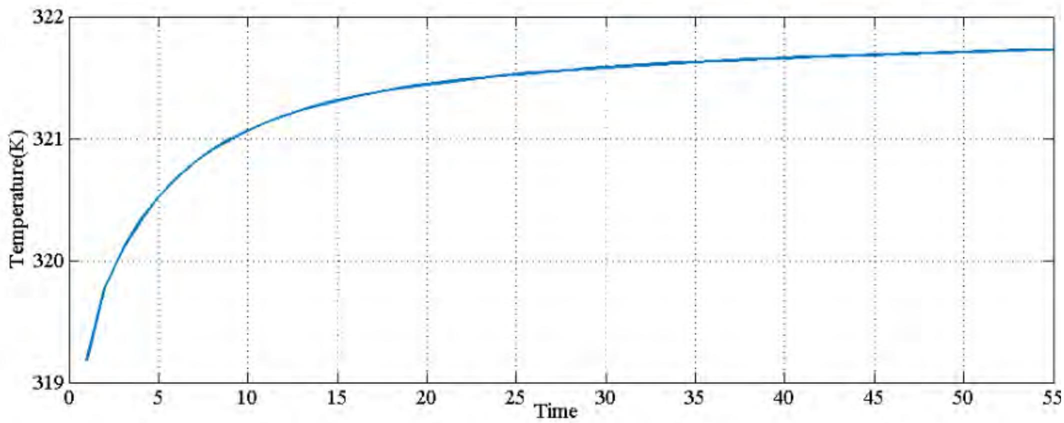

Figure 7 the Temperature Change of the Point Close to the Trickle

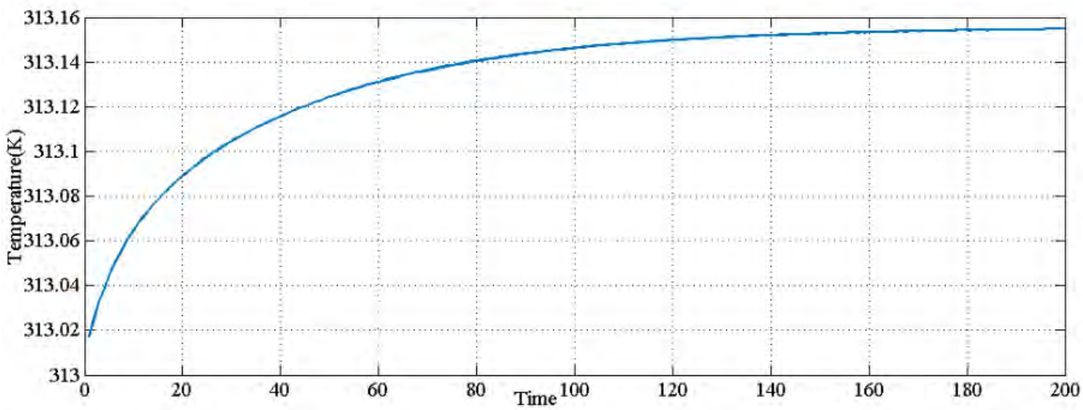

Figure 8 the Average Temperature Changing with Time

3.3 The stationary heat distribution

In this part, our model does not include man. It is the simplest model that we construct. We just want to know how the heat of bath water distributes in space. To avoid the uncertainty of heat distribution at different time, we only draw the graph of ultimate heat distribution in equilibrium state. In other words, the heat distribution is not changing with time when it reaches to equilibrium state.

So far, we have transformed a four-dimension problem into a three-dimension problem. However, there is still no perfect approach to describe a three-dimension object by graph. Additionally, we need to take each position's temperature into account. Thus we draw three views to present our results.

The front view, side view, top view are parallel to $y-z$ *plane*, $x-z$ *plane* and $x-y$ *plane*. The value of each point in the graph equals the average value of some points of the same characteristics in the solution domain. Take the front view for example, if two points' projections coincide with each other in $x-y$ *plane*, we deem they have the same characteristics in z direction.

The graph in figure 9 shows the heat distribution of bath water. It indicates the closer bath water to the trickle is, the higher temperature it will be. Red means a higher degree while yellow means a lower degree. And also the graph illustrate when heat exchange reachs a equalibrium, the heat distribution is uneven.

3.4 Optimal strategy

To illustrate, figure 9 and figure 10 are only different in the position of the faucet. Compared with Figure 9, Figure is more even in temperature. The result implies that if we change the position of the faucet where the trickle comes from, it will affect the heat distribution of bath water.

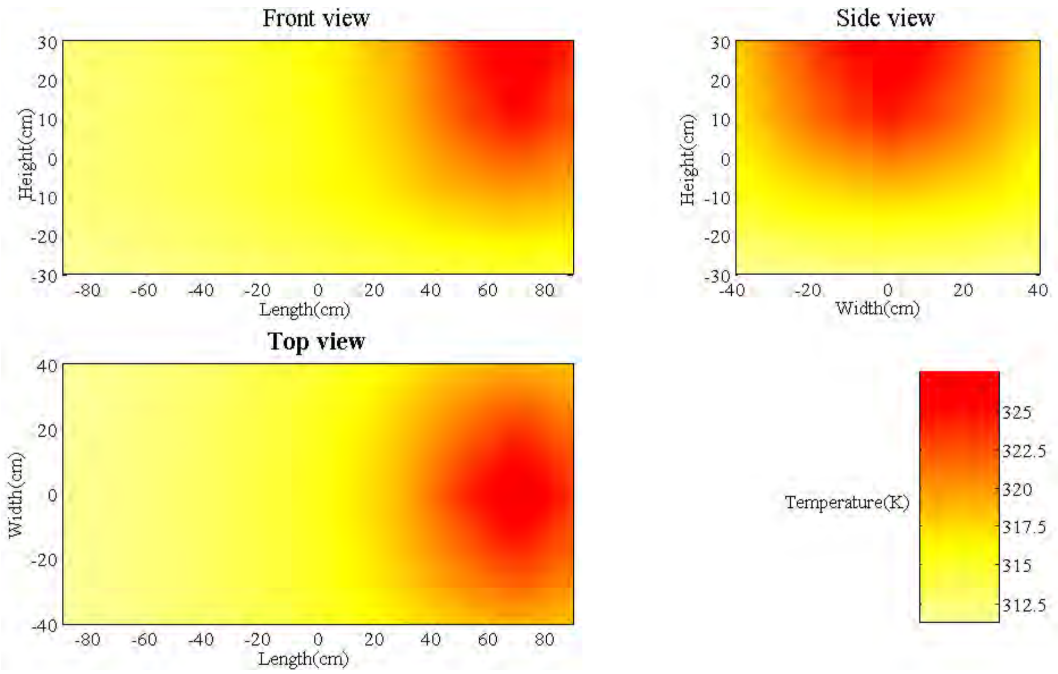

Figure 9 the Stationary Heat Distribution

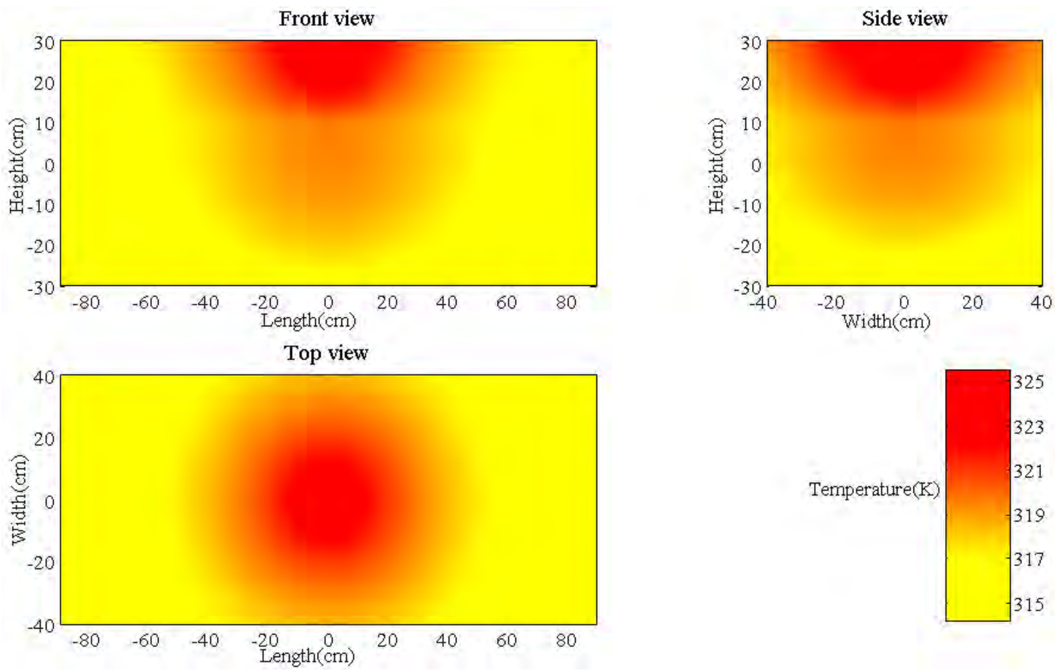

Figure 10 Heat Distribution of situation when faucet moves

We use computer simulation to determine which the best position that the trickle should enter through the surface water. And the answer is that the closer position to the center of the cuboid the trickle enters, the more homogeneous the heat distribution will be. Additionally, the closer position to the center of the cuboid the trickle enters, the less wasting of water there will be. So in a cuboid, "even" is always companied with "economic".

3.4.1 The definition of even degree

Question one asks us to develop a model of the temperature of the bathtub water in space and time to determine the best strategy that can keep the temperature even and close to the initial temperature with less wasting of water. But how to understand the word “even”? As far as we concerned, even in the equilibrium state, the temperature of different parts of bath water is not the same. There are two simple way to define the even degree. One is to get a smaller ΔT_{h-l} .

$$\Delta T_{h-l} = Min \quad \left(T_h - T_l\right) \tag{2-22}$$

where T_h is the highest temperature in bath water and T_l is the lowest temperature in bath water.

The other is a relatively complicated. We need to get a smaller σ^2 .

$$\sigma^2 = \frac{1}{I \times J \times K}\sum\sum\sum\left(T_{i,j,k} - \frac{\sum\sum\sum T_{i,j,k}}{I \times J \times K}\right)^2 \tag{2-23}$$

where $T_{i,j,k}$ is the temperature of points, σ is the standard derivation. If the bathtub is not in a cuboid shape, the more generalized formula (2-23) is

$$\sigma^2 = \lambda\sum\left(T - \bar{T}\right)^2 \tag{2-24}$$

where $\bar{T}$ is the average temperature, λ is a constant coefficient.

We use the variance to weigh the even degree. As shown in the figure 11, the value of minimum variance is not zero. Our results imply that the the perfect evenness can’t be realized.

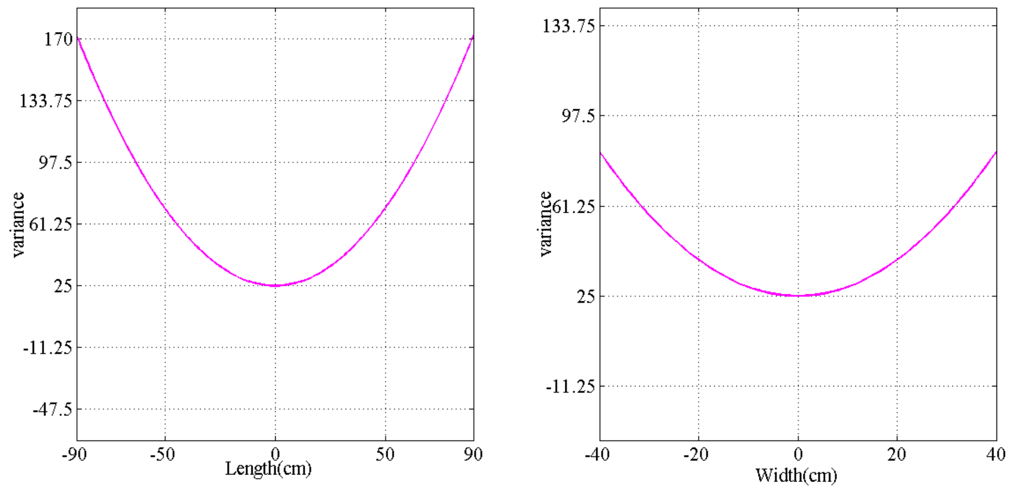

Figure 11 Variance Diagram

Based on our numerical solutions from computer simulation, keeping faucet's location near the center of the bathtub will contribute to make the heat distribution even. Theoretically, the heat distribution won't be in perfect uniform because the temperature of bath water near the hot trickle will be unavoidably higher than the temperature of that far away from the heat source. If possible, we'd better set the faucet at the center. The figure 12 shows the relationship between the equilibrium temperature and flows of the trickle.

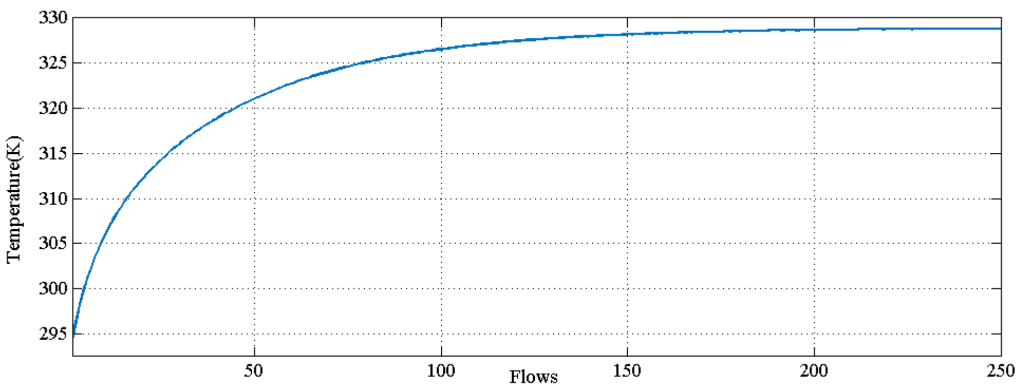

Figure 12 How the Temperature Influenced by Flows

3.4.2 Optimal temperature with less waste of water

Due to the notion of discretization, we assume the flow of water is proportional to the number of small fictitious heat source points. For any number of heat source points, we can determine the temperature in equilibrium state through computer simulation.

One common sense is that the temperature in equilibrium is not proportional to the flow of water. When the flow of water is small, changing the flow will cause more temperature change compared with larger flow. When the flow is extremely large but the influence of water flow turbulence can still be ignored, changing the flow will have few effects on the temperature in equilibrium. In other words, the total utility is suffering from loss. Therefore, the water is wasted.

According to computer simulation, we can draw a curve where the independent variable is the flow of trickle and the dependent variable is the temperature in equilibrium in a plane. Then according to this curve, we can determine the optimal flow of water so that we can avoid wasting too much water.

3.5 Situation when the volume or shape of bathtub changes

In this part, we attempt to understand how the volume of bathtub influence the heat distribution. We alter the volume of bathtub by changing the position of the outlet. In order to make analysis easier, we move the position of faucet and let the trickle through the center of the cuboid. Here, we don’t discuss the influence of man.

We graph our results. Figure 13 describes the condition when the volume of bathtub is small and figure 14depicts the situation when the volume of bathtub is big.

By comparison, we find the only difference between this two pictures is the color. When the total amount of water is small, it is easier to be heated. When both of two conditions reach the equilibrium. The temperature of situation one(figure 13) is higher than situation two(figure 14). The results also illustrate that less volume will save more energy.

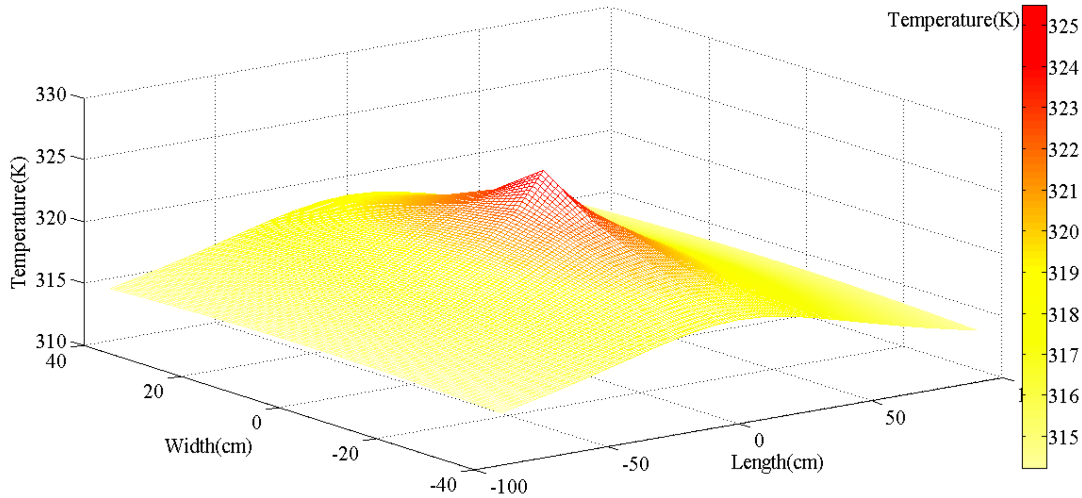

Figure 13 when the Volume of Bathtub is Small

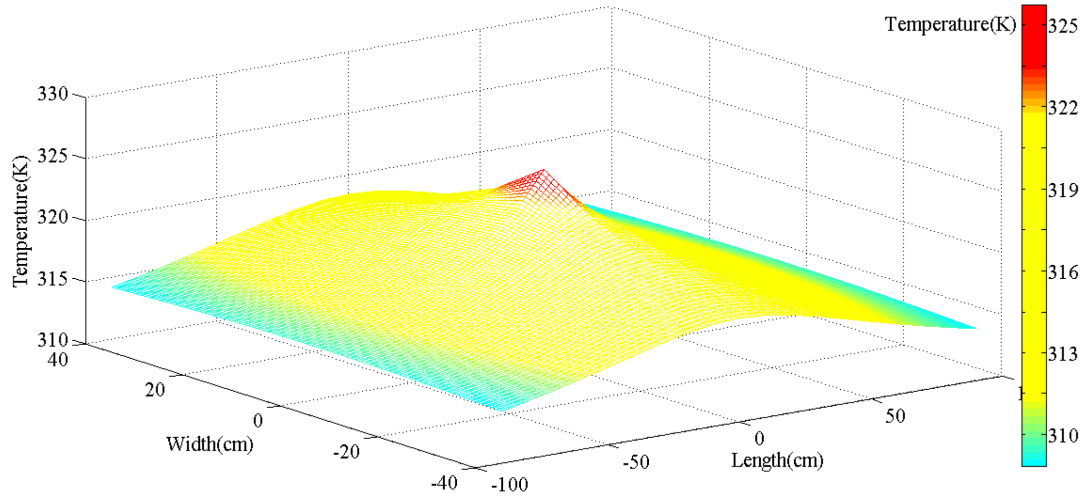

Figure 14 when the Volume of Bathtub is Big

3.6 Situation when people enters

In this part, we consider the condition when people enters. As we assumed, man can be viewed as a cuboid with constant temperature. Then we calculate the heat distribution and the results are shown in figure 15. The blue area can approximately depict where the man sit.

We find there is a clear bright yellow curve around the blue area. That’s because the temperature of man’s skin is lower than the water alongside. Heat transfer occurs in the junction of skin and bath water. So this part of water’s temperature is obviously lower than the rest.

This finding is corresponding to our life experience. Assume that you fill a washbasin with hot water and are ready for a foot bath. If you put your feet into the water and keep stationary. Then you feel the water alongside your feet is gradually losing heat and decrease in temperature.

If you move your feet, you will feel a sense of warmth. You find it interesting that water becomes hot again. Actually, it is because the uneven heat distribution in the water. The water alongside your feet is cooler, because it is transferring heat to your feet.

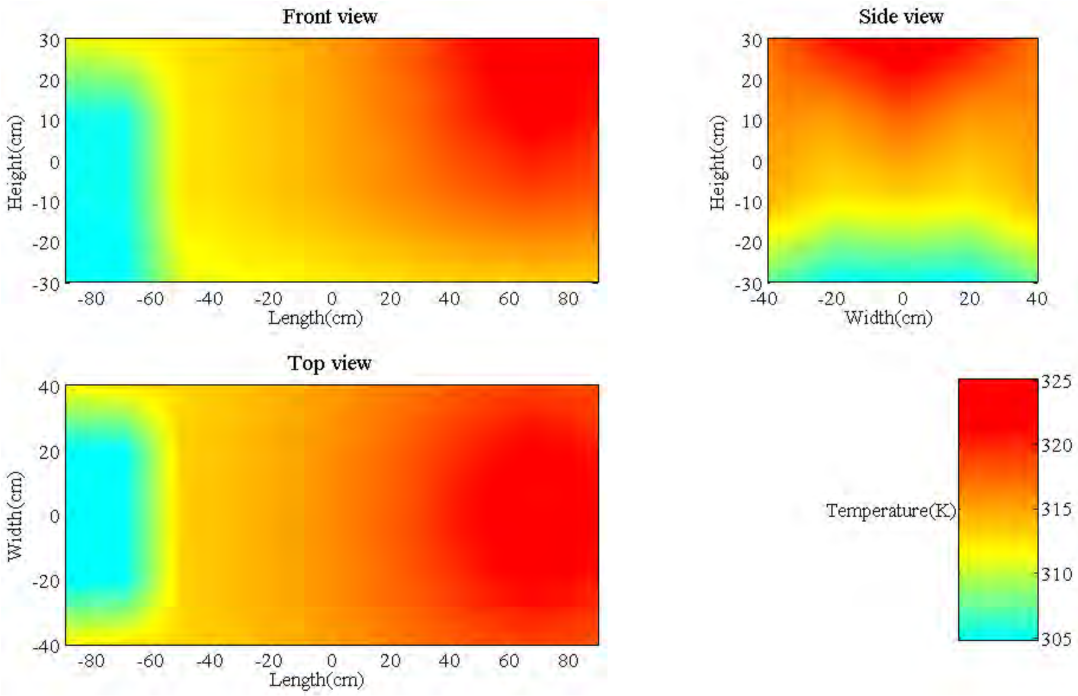

Figure 15 when People Enters

3.7 Situation when the volume of people changes

In order to capture the influence caused by different size of people, we draw figure 16. Figure 15 describes the heat distribution when a thin man enters and Figure 16 tell us to what extent a heavy man with a larger size will impact the heat distribution.

As can be seen, the lager the person's volume is, the smaller the volume of water in the bathtub is. As the person's volume increasing, the contact area between the person and bathtub increases but the contact area between the bathtub and water decreases.

Compared with figure 15, one of the most differences in figure 16 are the blue area is sharply enlarged. What's more, the bright yellow belt is wider. We can draw a conclusion that man's volume has a great impact on the heat distribution.

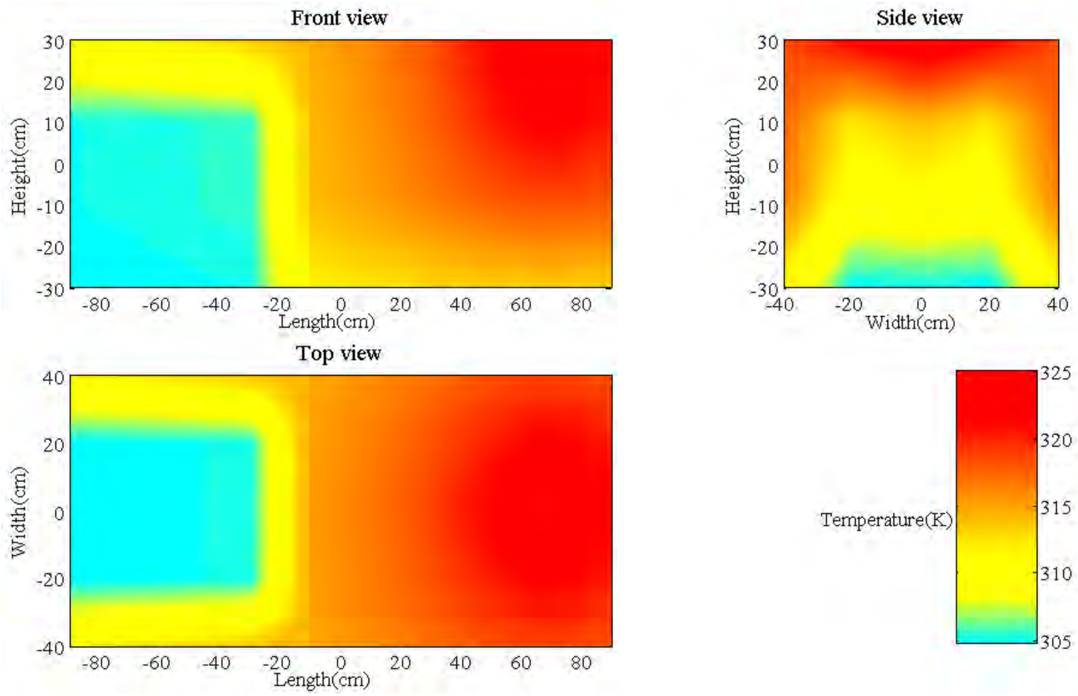

Figure 16 Condition when the Volume of People Increases

3.8 Situation when the shape of people changes

To depict the influence caused by different shape of people, we draw figure 17. Figure 17 is a three-dimension graph from top view. From figure 17-a, 17-c and 17-d. The cuboid in different length will influence the heat distribution. The shorter the cuboid is, the slower the change of temperature is. Moreover, as for the yellow bright belt alongside the blue area, there is no obvious evidence that can illustrate man’s shape will affect the speed of heat exchange between shell and water.

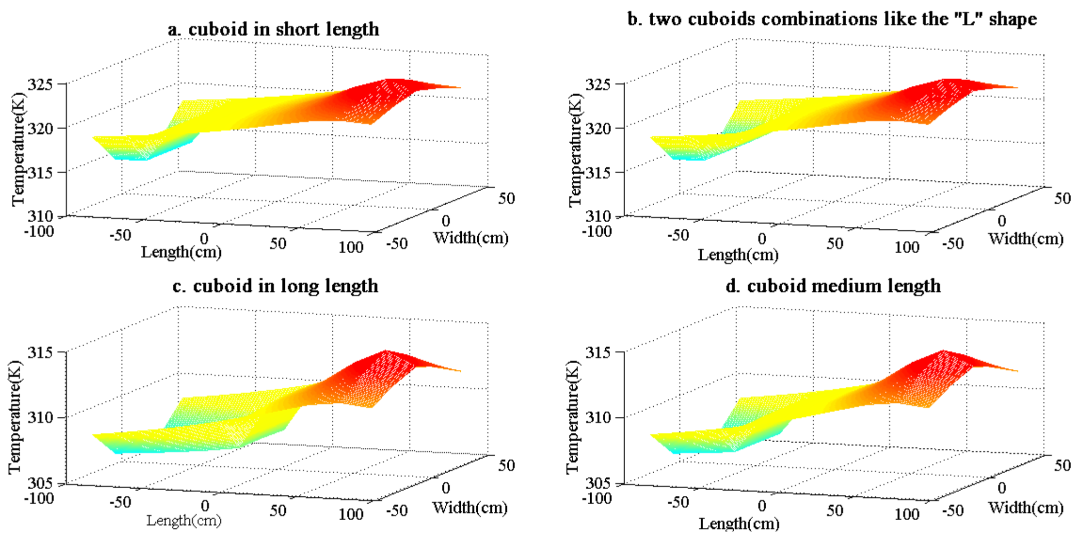

Figure 17 when the Shape of People Changes

3.9 Situation when the temperature of people changes

Based on our own understanding, there is no much meaning to discuss the temperature of people changes. Although different parts of man's body are not in the same temperature, man's shell temperature is always around 37 degree centigrade. Generally speaking, the temperature of man's shell ranges from 36.2 to 37.2 degree centigrade.

The results are just as what we expected before. Figure 18 is the result when the temperature of people changes. Compared with figured 15, we set man's shell temperature 1 degree centigrade lower. The heat distribution of these two figures are very alike.

Therefore, we can come to a conclusion that the temperature alongside the man's shell is not sensitive to shell temperature.

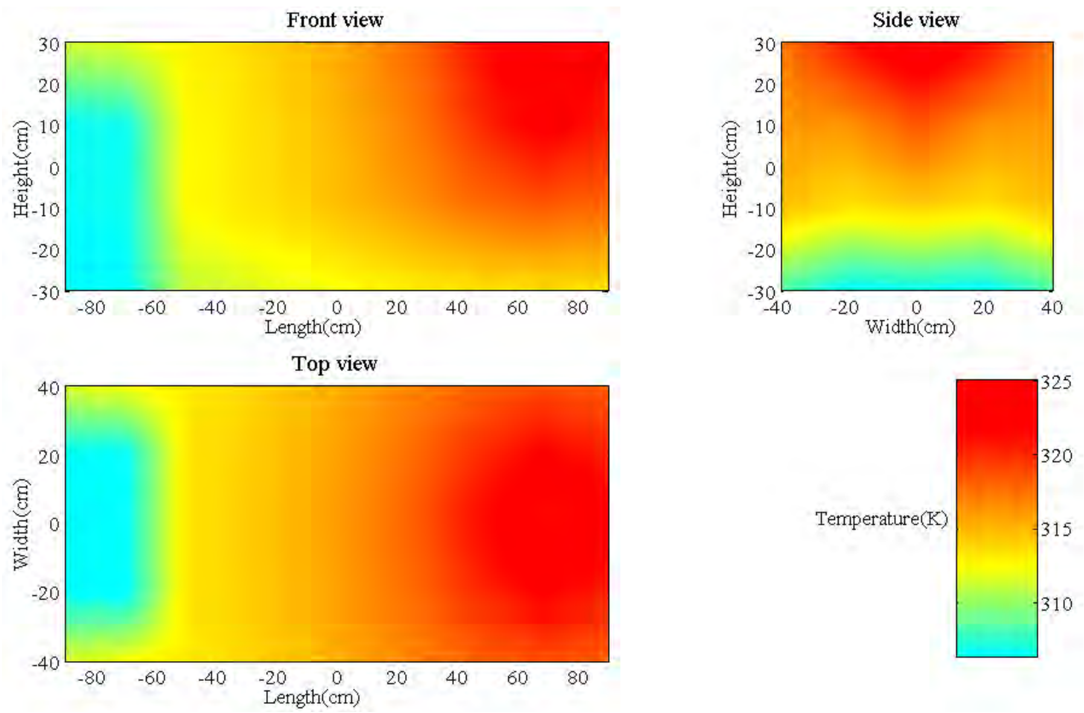

Figure 18 when the Temperature of People is Lower

4 Sensitivity analysis

In this section, we implement sensitivity analysis for our model. Specifically, we test the sensitivity of ambient temperature, trickle temperature

We analysis the effect of the two factors on equilibrium temperature.

4.1 ambient temperature

Figure 19 indicates that when ambient temperature changes from $283K$ to $303K$, the ultimate equilibrium temperature changes. And they are in positive correlation. When the ambient temperature is low, before the heat tranfering to it the point loses its own heat more quickly. So that the slope correspoing to the lower ambient at the very beginning is steeper, and the equilibrium temperature is lower.

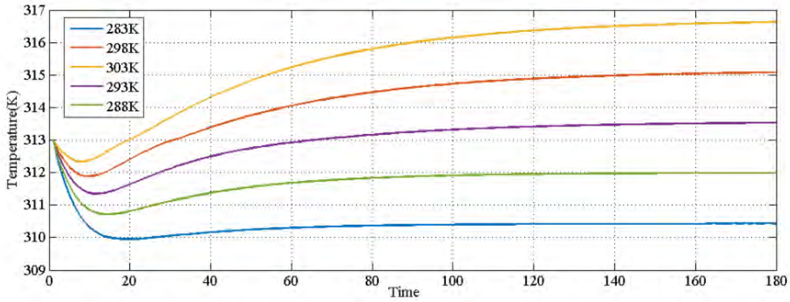

Figure 19 the Equilibrium Temperature with Different Ambient Temperature

4.2 the temperature of trickle

Figure 20 illustrates that when trickle temperature changes from $313K$ to $333K$, the ultimate equilibrium temperature changes correspondently. And they are in positive correlation. On all conditions, the point experiences a process of cooling. Where the trickle temperature is closest to the initial water temperature, the equilibrium state will be reached in advance.

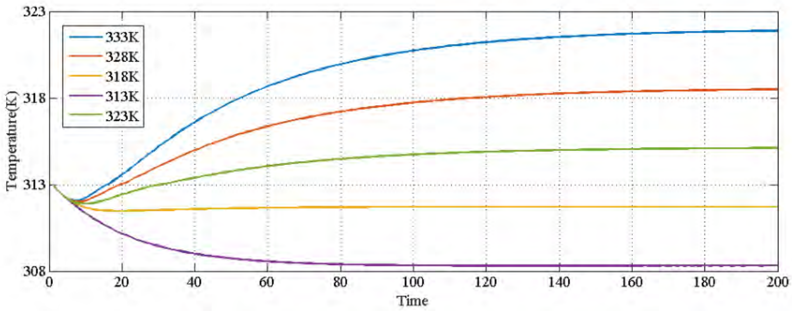

Figure 20 the Equilibrium Temperature with Different Water Temperature

4.3 the material of the bathtub

Here we discuss the situation when the material of bathtub changes. Different kinds of materials are varies in heat transfer coefficient. The higher heat transfer coefficient is, the lower equilibrium temperature will be.

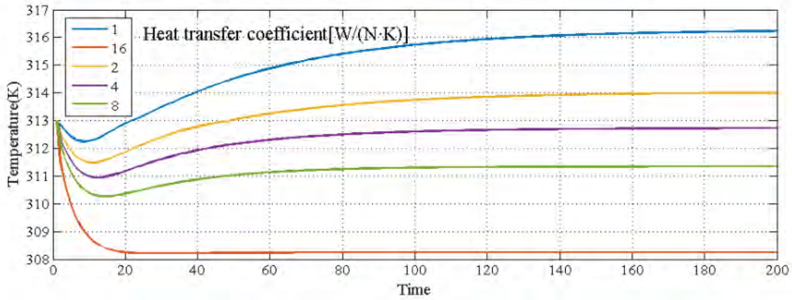

Figure 21 the Equilibrium Temperature with the Material of the Bathtub

5 Strengths and weaknesses

5.1 Strengths

- We use computer simulation which is a new developing method recently to calculate the heat distribution for us. It calculating speed is very quick and output can be continuous. Our computer simulation describes the real process of heat conduction and present the final graph of heat distribution at the equilibrium.
- In this passage, we need to adjust a lot of parameters for there are abundant variables and our way of solving the problem makes it very easier. We tell the computer what it should do and then it will output a series of results that we demand.
- We don’t need to solve a problem in four dimension any more. Use the numerical solutions to supplant the analytic solutions is a wise choice in these problems. We have fully made use of the strengths of taking the numerical solutions. With the computer and programming available, we can get the numerical solutions right after we input the order to the computer.
- We solve the problem in a micro way, which help us construct the model simpler and easier to understand.

5.2 Weaknesses

- The computational accuracy might be far from satisfaction and the error exists
- Due to the computational complexity, it may take us some time before results are calculated. Sometimes the running time of the program is very long.

5.3 Future work

The modelling framework used in this analysis provides scope for future enhancement. The following tasks will reduce the assumptions made in this paper so as to produce a better model of actual heat transfer and distribution.

- Obtaining necessary date to accurate the setting of parameters by experiment, we can promote a great realistic meanings of our results.
- According to the real size and volume of bathtub, use computer simulation to get a vivid graph of heat contribution and determine which strategy is the best.
- Increasing the scope of sensitivity analysis to cover all possible parameter changers.
- Adding additional variables that might change the heat distribution to our model and analysis their effects.

6 Letter to Users of the Bathtub

To whom it may concern,

We have exerted ourselves to design a bathtub that can both physically and economically meet your demand. However, the results are still far from satisfactory. We must recognize a fact that there is no perfect scientific theory that can be well applied to our research.

We simulate the temperature change of hot bath water by applying our simplified model. As long as the size of the bathtub, the volume of the person, the initial temperature of the water, the trickle’s temperature and some necessary conditions are given, we can determine the best strategy including the location where the hot water enters the bathtub, the flow of the water from the faucet, etc.

Additionally, with given conditions above, we calculate and graph the heat distribution of bath water. To search and test the best strategy’s performance, we change the value of certain variable while keeping other variables constant. The results illustrate that the superficial area of bath water is one of the most effective factors that alter the ultimate equilibrium temperature. The influence caused by the shape of bathtub matters when it changes the superficial area of bath water. The equilibrium temperature is very sensitive to man’s volume. The change of the equilibrium temperature altered by man’s regular motion can be ignored to some extent. However, man’s motion contributes greatly to even temperature distribution.

It means when we determine the best strategy we must consider the influence of superficial area of bath water.

We assumed that some sets of conditions which will be possibly confronted with in our real life. Based on our numerical solutions from computer simulation, keeping faucet's location near the center of the bathtub will contribute to make the heat distribution even. Theoretically, the heat distribution won't be in perfect uniform because the temperature of bath water near the hot trickle will be unavoidably higher than the temperature of that far away from the heat source. If possible, we'd better set the faucet at the center. Additionally, taking a bubble bath will help to reduce the heat loss.

In conclusion, the results indicated that to achieve an evenly maintained temperature throughout the bath water is almost impossible. What's more, the reality is much more complex and unpredictable than our hypothetic problem-space. That's why the perfect "evenness" is so hard to achieve.

Yours sincerely

MCM Team

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Appendix

the realization of our own computer simulation system

```
//object.h
//The definition of data structure and functions
#ifndef _OBJECT_H
#define _OBJECT_H
#define XBOUNDARY 180
#define YBOUNDARY 80
#define ZBOUNDARY 60
typedef struct{
    int type;
    float t;
    float heatresistance;
}vertex_type;
extern vertex_type vertex[XBOUNDARY][YBOUNDARY][ZBOUNDARY];
extern void init();
extern void setHearSource(int x1,int x2,int y1 ,int y2,int depth,float temp);
extern void simulation(float R1,float R2,float R3);
#endif

//object.cpp
//The implement of the main algorithm
#include<iostream>
#include<fstream>
#include "object.h"
```

```
#define NOTHEATSOURCE 0
#define HEATSOURCE 1
#define BEGINTEMP 40
#define environmentTemp 25
using namespace std;
vertex_type vertex[XBOUNDARY][YBOUNDARY][ZBOUNDARY];
//Initialization
void init()
{
	for(int x=0;x<XBOUNDARY;x++){
		for(int y=0;y<YBOUNDARY;y++){
			for(int z=0;z<ZBOUNDARY;z++){
				vertex[x][y][z].type=NOTHEATSOURCE;
				vertex[x][y][z].t=BEGINTEMP;
			}
		}
	}
}
//The setting of heat source
void setHearSource(int x1,int x2,int y1 ,int y2,int depth,float temp)
{
	for(int x=x1;x<=x2;x++){
		for(int y=y1;y<=y2;y++){
			for(int i=0;i<depth;i++){
				vertex[x][y][ZBOUNDARY-i].type=HEATSOURCE;
				vertex[x][y][ZBOUNDARY-i].t=temp;
			}
		}
	}
}
void simulation(float R1,float R2,float R3)
{
	ofstream fout("temp.txt");
	float dt,trant,average;
	double sum1=0,sum2=0;
	int count=0;
	while(true){
		float t=vertex[0][0][0].t;
		/*for(int x=0;x<XBOUNDARY;x++){
			for(int y=0;y<YBOUNDARY;y++){
				for(int z=0;z<ZBOUNDARY;z++){
					sum1+=vertex[x][y][z].t;
				}
			}
		}*/
		for(int x=0;x<XBOUNDARY;x++){
			for(int y=0;y<YBOUNDARY;y++){
				for(int z=0;z<ZBOUNDARY;z++){
					if(x+1!=XBOUNDARY ){
						if(vertex[x][y][z].t>vertex[x+1][y][z].t){
```

```
            dt=vertex[x][y][z].t-vertex[x+1][y][z].t;
            trant=dt/R1;
            if(vertex[x][y][z].type==0){
                vertex[x][y][z].t=vertex[x][y][z].t-trant;
            }
            vertex[x+1][y][z].t=vertex[x+1][y][z].t+trant;
        }
    }
    if(x-1!=-1 ){
        if(vertex[x][y][z].t>vertex[x-1][y][z].t){
            dt=vertex[x][y][z].t-vertex[x-1][y][z].t;
            trant=dt/R1;
            if(vertex[x][y][z].type==0){
                vertex[x][y][z].t=vertex[x][y][z].t-trant;
            }
            vertex[x-1][y][z].t=vertex[x-1][y][z].t+trant;
        }
    }
    if(y+1!=YBOUNDARY ){
        if(vertex[x][y][z].t>vertex[x][y+1][z].t){
            dt=vertex[x][y][z].t-vertex[x][y+1][z].t;
            trant=dt/R1;
            if(vertex[x][y][z].type==0){
                vertex[x][y][z].t=vertex[x][y][z].t-trant;
            }
            vertex[x][y+1][z].t=vertex[x][y+1][z].t+trant;
        }
    }
    if(y-1!=-1 ){
        if(vertex[x][y][z].t>vertex[x][y-1][z].t){
            dt=vertex[x][y][z].t-vertex[x][y-1][z].t;
            trant=dt/R1;
            if(vertex[x][y][z].type==0){
                vertex[x][y][z].t=vertex[x][y][z].t-trant;
            }
            vertex[x][y-1][z].t=vertex[x][y-1][z].t+trant;
        }
    }
    if(z+1!=ZBOUNDARY ){
        if(vertex[x][y][z].t>vertex[x][y][z+1].t){
            dt=vertex[x][y][z].t-vertex[x][y][z+1].t;
            trant=dt/R1;
            if(vertex[x][y][z].type==0){
                vertex[x][y][z].t=vertex[x][y][z].t-trant;
            }
            vertex[x][y][z+1].t=vertex[x][y][z+1].t+trant;
        }
    }
    if(z-1!=-1 ){
        if(vertex[x][y][z].t>vertex[x][y][z-1].t){
```

```
                        dt=vertex[x][y][z].t-vertex[x][y][z-1].t;
                        trant=dt/R1;
                        if(vertex[x][y][z].type==0){
                            vertex[x][y][z].t=vertex[x][y][z].t-trant;
                        }
                        vertex[x][y][z-1].t=vertex[x][y][z-1].t+trant;
                    }
                }
                if(x==XBOUNDARY||x==0){
                    dt=vertex[x][y][z].t-environmentTemp;
                    trant=dt/R2;
                    if(vertex[x][y][z].type==0){
                        vertex[x][y][z].t=vertex[x][y][z].t-trant;
                    }
                }
                if(y==YBOUNDARY||y==0){
                    dt=vertex[x][y][z].t-environmentTemp;
                    trant=dt/R2;
                    if(vertex[x][y][z].type==0){
                        vertex[x][y][z].t=vertex[x][y][z].t-trant;
                    }
                }
                if(z=ZBOUNDARY){
                    dt=vertex[x][y][z].t-environmentTemp;
                    trant=dt/R3;
                    if(vertex[x][y][z].type==0){
                        vertex[x][y][z].t=vertex[x][y][z].t-trant;
                    }
                }
                if(z==0){
                    dt=vertex[x][y][z].t-environmentTemp;
                    trant=dt/R2;
                    if(vertex[x][y][z].type==0){
                        vertex[x][y][z].t=vertex[x][y][z].t-trant;
                    }
                }
            }
        }
    }
/*  for(int x=0;x<XBOUNDARY;x++){
        for(int y=0;y<YBOUNDARY;y++){
            for(int z=0;z<ZBOUNDARY;z++){
                sum2+=vertex[x][y][z].t;
            }
        }
    }*/
/*  if(sum1-sum2<0.0000000000001){
        for(int x=0;x<XBOUNDARY;x++){
            for(int y=0;y<YBOUNDARY;y++){
                for(int z=0;z<ZBOUNDARY;z++){
```

```
                    cout<<vertex[x][y][z].t;
                }
            }
        }
    }*/
    count++;
    if(count==1000){
    //  average=sum1/(180*60*80);
        fout<<vertex[0][0][0].t<<endl;
    //  fout<<average<<endl;
        count=0;
    }
    sum1=0;
  }
}

//main.cpp
#include "object.h"
int main()
{
  init();
  setHearSource(154,159,38,40,20,50);
  simulation(3,2000,100000);
}
```